Effect of the pulse trajectory on ultrasonic fluid velocity measurement

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Abstract In a general situation a non-uniform velocity field gives rise to a shift of the otherwise straight acoustic pulse trajectory between the transmitter and receiver transducers of a sonic anemometer. The aim of this paper is to determine the effects of trajectory shifts on the velocity as measured by the sonic anemometer. This determination has been accomplished by developing a mathematical model of the measuring process carried out by sonic anemometers; a model which includes the non-straight trajectory effect. The problem is solved by small perturbation techniques, based on the relevant small parameter of the problem, the Mach number of the reference flow, M. As part of the solution, a general analytical expression for the deviations of the computed measured speed from the nominal speed has been obtained. The correction terms of both the transit time and of the measured speed are of M^2 order in rotational velocity field. The method has been applied to three simple, paradigmatic flows: one-directional horizontal and vertical shear flows, and mixed with a uniform horizontal flow.

List of symbols

c	sound speed
C_2	coefficient of the measured velocity correction term
C_{20}	part of C_2 due to asymptotic expansion
C_{2s}	part of C_2 due to take into account the deviation of the ray path from a straight line
l	length of measurement path
M	Mach number
$\mathbf{n}$	unit vector normal to the wave front
t_+, t_-	transit times in forward an backward directions
T_+, T_-	dimensionless transit times in forward an backward directions
u, w	fluid's velocity components
u_P, w_P	total wave propagation speed components
U, W	dimensionless fluid's velocity components
u_M	measured velocity along the measurement path
U_M	dimensionless measured velocity along the measurement path
$\mathbf{v_P}$	total wave propagation speed
$\mathbf{v_W}$	fluid's velocity
v_R	reference velocity
X, Z	dimensionless coordinates
z_{F+}, z_{F-}	forward and backward pulse trajectory expressed in a moving reference frame
z_+, z_-	forward and backward pulse trajectory expressed in a fixed reference frame
Z_+, Z_-	deviation of the ray path from a straight line in forward and backward directions
Z_1,	coefficient of M order term in the expressions of Z_+, Z_-
Z_2	coefficient of M^2 order term in the expressions of Z_+, Z_-

1 Introduction

The aim of this paper is to analyze one of the sources of uncertainty in the measurement of wind speed by using ultrasonic anemometers: the shift of the trajectory of the ultrasonic pulse from the straight path caused by the velocity field being measured. With this aim, a mathematical model of the physical process of the ultrasound signal propagation between the transmitter and the receiver has been developed which takes into account the trajectory drift from the straight path produced by the velocity field, unlike the state-of –the –art models which consider just the straight path propagation (Kaimal et al. 1968; Silverman 1968; Kristensen and Fitzjarrald 1984; Cuerva et al. 2003).

Ultrasonic anemometry is a technique that measures the wind speed vector based on the detection of the influence of the flow field on the transmission of ultrasonic signals between a transmitter and a receiver. A pair of facing transducers defines a measurement path. This method allows the velocity vector component to be parallel to the measurement path. The simplest configuration of a measurement path placed inside a uniform parallel flow, u_∞ is shown in Fig. 1. The transducers are placed facing each other and separated a distance l.

Each transducer emits an ultrasound signal that travels towards the opposite transducer. The speed of propagation of the signals is the vector sum of the speed of sound in the media, c, and the local flow velocity vector along the measurement path. Therefore, the flow velocity field has a different effect on the propagation of signals in each direction. From the measurement of the differences between the forward and backward transit times, the wind speed component along the measurement path can be obtained.

If the measurement path of a sonic anemometer is modelled by the geometrical arrangement shown in Fig. 1, the transit times in each direction could be related with the flow speed, u_∞, and the speed of sound by

$$t_+ = \frac{l}{c + u_\infty}, \quad t_- = \frac{l}{c - u_\infty}, \tag{1}$$

where the + subscript refers to the transit time of the signal travelling from P1 to P2 and the – subscript to the signal travelling in the opposite direction. The anemometer measures both transit times and determines the flow velocity using the following expression deduced from Eq. 1:

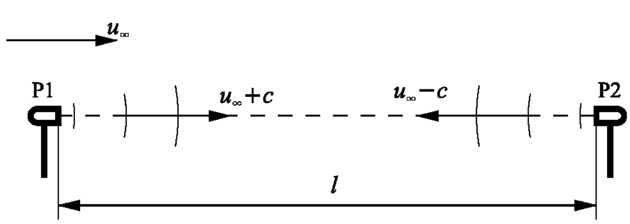

Fig. 1 Basic configuration of a measurement path, whose length is l, placed in an uniform flow u_∞, P1 and P2 are the ultrasonic transducers. c is the sound speed

$$u_M = u_\infty = \frac{l}{2}\left(\frac{1}{t_+} - \frac{1}{t_-}\right); \tag{2}$$

This is the algorithm being used by most ultrasonic anemometers nowadays, regardless of the velocity field. This expression, however, is valid only if the velocity field is uniform.

The mathematical model presented here gives expressions of the transit-time of the ultrasound signal from the transmitter to the receiver in both directions, taking into account the effect of the velocity field along the measurement path, the so called "nominal velocity field" (Franchini 2006). From these expressions the measured flow speed, as given by the anemometer, is obtained by applying the time inverse difference algorithm (Eq. 2).

The nominal velocity field employed in the model is defined by a set of parameters; therefore, the expression of the measured velocity thus obtained also includes in some way these parameters which allow analysis of their influence. The result is compared with the nominal velocity component that is intended to be measured. By doing so, the deviation of the measured speed from the nominal velocity, due to the measurement process, is obtained.

This method is applied in another paper (Franchini et al. 2007) to study the deviations associated with the measurement the flow rate inside a pipe. This method could also be employed to study the effect of the perturbations of the sensor structural supports of ultrasonic anemometers.

This paper is organized as follows. In Sect. 2, a brief overview of acoustic theory fundamentals is presented, and the relevant hypothesis considered. In Sect. 3 the formulation of the problem is described, the drift of the acoustic signal from the straight path between the sensors is obtained, and its impact on the transit time and on the velocity measured according to the inverse time algorithm analysis. In Sects. 4 and 5 an approximate solution of the problem is obtained by using asymptotic series expansions based on Mach number powers, taking into consideration small deviations of the pulse trajectory from the straight line between transducers. This leads to analytical expressions of the corrections of the measured speed. In order to make clear the process and to show the manner in which the flow characteristics affect the determination of speed, the method is applied to three elementary and paradigmatic flow models (Sect. 6). The main conclusions are summarized in Sect. 7.

2 Fundamentals of acoustics

Under the usual assumptions of the acoustic theory, the propagation of acoustic waves obeys the wave equation (Landau and Lifshitz 1989 and Pierce 1991)

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \tag{3}$$

where p is the pressure perturbation. The geometric acoustic approach can be applied to the ultrasonic anemometer configurations as it is explained in the following paragraph. In this approach the parts of the monochromatic acoustic wave with the same phase is considered to form an imaginary moving surface called wave-front. If this monochromatic acoustic wave propagates in such a way that the amplitude and propagation direction varies in small amounts at distances along the wavelength, it is possible to ignore the wave characteristic of sound and consider that the transmission occurs along lines; the so called acoustic rays, whose tangents at each point have the direction of propagation and are perpendicular to the wave front.

If the wave propagates itself in a fluid moving at speed $\mathbf{v_W}$, the total wave propagation speed with regard to a fixed reference system, $\mathbf{v_P}$, is

$$\mathbf{v_P} = \mathbf{v_W} + c\mathbf{n} \tag{4}$$

where $\mathbf{n}$ is the unit vector normal to the wave front (Fig. 2) which follows the propagation direction with regard to a reference system moving with the fluid.

The evolution of the position vector of a wave front point, $\mathbf{x_P}(t)$, in a fixed reference frame is given by

$$\frac{\mathrm{d}\mathbf{x_P}}{\mathrm{d}t} = \mathbf{v_W}(\mathbf{x_P}, t) + c(\mathbf{x_P}, t)\mathbf{n}(\mathbf{x_P}, t) \tag{5}$$

where in general, both the wind speed and the speed of sound depend on both position and time. Assuming that the sound speed is constant and $|\mathbf{v_W}| \ll c$, Landau and Lifshitz (1989) showed that at first order

$$\frac{\mathrm{d}\mathbf{n}'}{\mathrm{d}s} = -\mathbf{n}' \times \frac{(\nabla \times \mathbf{v_W})}{c} \tag{6}$$

where s is the length measured along the trajectory, $\mathbf{n}'$ is tangent to the trajectory, and $\mathrm{d}\mathbf{n}'/\mathrm{d}s$ is the local curvature of the ray trajectory.

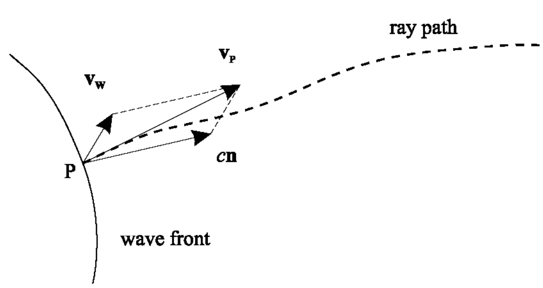

Fig. 2 Propagation of a wave front in a moving fluid. Concept of ray trajectory. $\mathbf{V_W}$ is the fluid velocity. c is the sound speed. $\mathbf{V_P}$ is the total pulse speed along the ray trajectory. $\mathbf{n}$ is the direction of propagation as seen in a reference frame moving with the fluid

According to this expression, the existence of some curvature of the trajectory requires that the rotational of the velocity field be different from zero, and in such a case the curvature would be of the order of the Mach number, $M = |\mathbf{v_W}|/c$.

In an ultrasonic anemometer, the frequency of the acoustic wave emitted by the transducers is of the order of at least 4×10^4 Hz (Mylvaganam 1989, O'Sullivan and Wright 2002). Therefore, it is possible to consider the wavelength $\lambda_U \ll l$ and to apply the concepts derived from the geometrical acoustics approach.

3 Formulation of the problem

Let us consider the trajectory followed by an acoustic signal from the emitter, P1 to the receiver, P2, which depends on the velocity field present along the trajectory, as shown in Fig. 3. The trajectory considered is contained in the x–z plane defined by two vectors: the direction of the measurement path (P1–P2) and the mean wind speed vector. It is assumed that the normal speed to this plane is negligible (defined as its mean value being zero).

The transit time measurement starts when the acoustic pulse is emitted by the transducer P1 and finishes when the transducer placed at P2 detects the first arriving wave. Therefore, the problem consists of determining the minimum transit time trajectory between P1 and P2. This is the trajectory that minimizes the value of the integral

$$t_+ = \int_{x_1}^{x_2} \frac{\mathrm{d}x}{u_P(x, z, t)} \tag{7}$$

where $u_P(x, z, t)$ is the total velocity component of the acoustic pulse along the x-axis.

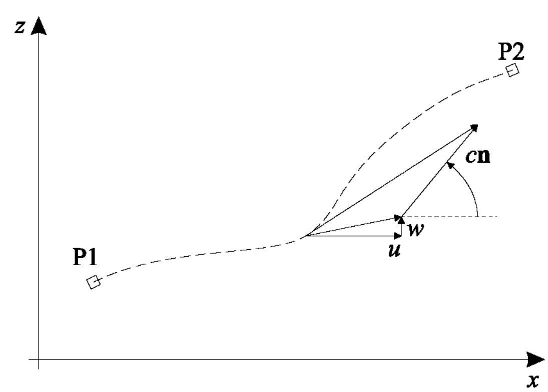

Fig. 3 Sketch of the ultrasound pulse trajectory from the emitter P1 to the receiver P2. Speed diagram: u,w, fluid velocity components. c: sound speed. $\mathbf{V_W}$, $\mathbf{V_P}$: wind velocity and total pulse velocity in the fixed reference frame

A similar expression applies to the acoustic pulse travelling in the opposite direction, from P2 to P1. Let us analyze first the propagation from P1 to P2. The fluid properties are considered to be uniform and therefore the sound speed is constant along the measurement path. In expression (5) the direction of the acoustic ray $\mathbf{n}$ (Fig. 3) and the wind speed can be written as

$$\mathbf{n}(x,z,t) = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j} \tag{8}$$

$$\mathbf{v_W} = u(x,z,t)\mathbf{i} + w(x,z,t)\mathbf{j} \tag{9}$$

where α is the angle of the wave front propagation direction in the reference system moving with the fluid, and $u(x,z,t)$ and $w(x,z,t)$ are the components of the wind velocity, respectively.

The trajectory of the pulse, $z(x,t)$, that minimizes Eq. 7 is given by the solution of the Euler equation (Forray 1968)

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'} - \frac{\partial f}{\partial z} = 0, \tag{10}$$

together with the boundary conditions $z(x_1) = z_1, z(x_2) = z_2$, where $f = u_P(x,z,z',t)^{-1}$. To find the solution to the problem, the total velocity $u_P(x,z,t)$ is written as a function of the trajectory and the wind speed components. The pulse speed components in the anemometer fixed reference frame are:

$$u_P = \frac{\mathrm{d}x}{\mathrm{d}t} = u(x,z,t) + c\cos\alpha \tag{11}$$

$$w_P = \frac{\mathrm{d}z}{\mathrm{d}t} = w(x,z,t) + c\sin\alpha. \tag{12}$$

where

$$\alpha = \arctan z'_{F+} \tag{13}$$

and z'_F is the tangent of the angle between the x-axis and pulse propagation direction in the reference frame moving with the fluid. So, z_F can be identified as the pulse trajectory in the reference frame moving with the fluid. Therefore the total pulse velocity is,

$$u_P = \frac{\mathrm{d}x}{\mathrm{d}t} = u(x,z_+,t) + \frac{c}{\sqrt{1+z'^2_{F+}}}, \tag{14}$$

$$w_P = \frac{\mathrm{d}z_+}{\mathrm{d}t} = w(x,z_+,t) + c\frac{z'^2_{F+}}{\sqrt{1+z'^2_{F+}}}. \tag{15}$$

The transit time becomes

$$t_+ = \int_{x_1}^{x_2} \frac{1}{u+\frac{c}{\sqrt{1+z'^2_{F+}}}}\mathrm{d}x = \frac{l}{c}\int_{X_1}^{X_2} \frac{1}{\frac{u}{c}+\frac{1}{\sqrt{1+Z'^2_{F+}}}}\mathrm{d}X \tag{16}$$

where $u = u(x,z,t)$, $x = l\,X$, $z_{F+} = l\,Z_{F+}$ and l is the path length. The dimensionless transit time is obtained from (16)

$$T_+ = t_+\frac{c}{l} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}X}{UM+\frac{1}{\sqrt{1+Z'^2_{F+}}}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} F_+\mathrm{d}X \tag{17}$$

where

$$F_+ = \left(UM + \frac{1}{\sqrt{1+Z'^2_{F+}}}\right)^{-1} \tag{18}$$

by using dimensionless velocity components

$$U = \frac{u}{v_R}, \quad W = \frac{w}{v_R} \tag{19}$$

where v_R is a reference velocity defined in a case-by-case basis for convenience, and $M = v_R/c$ is the Mach number. It has been assumed also that both sensors P1 and P2 are placed on the x axis, symmetric with regard to the origin (P1(−1/2,0,0), P2(1/2,0,0)), as shown in Fig. 4. $F_+ = F_+(X, Z, t)$ is the function to be substituted in the Euler equation (Eq. 10) instead of f, in order to determine the value of the pulse trajectory slope in moving axes Z'_{F+} that gives rise to the minimum time trajectory.

4 Solution by asymptotic series expansion

To find the function $Z_{F+}(x)$ which is the solution to problem (10), the method of asymptotic series expansions has been used. As the wind velocity is small compared to the sound speed the Mach number is $M \ll 1$, it can be considered as a small parameter for this problem and therefore, the lateral shifts $Z_+(x)$ and its derivatives are also small. Furthermore, no variation of the wind speed during the measuring time (steady flow field) is considered, as derived from the condition $t_c \ll l/c$ where t_c is the characteristic time of variation of the velocity field.

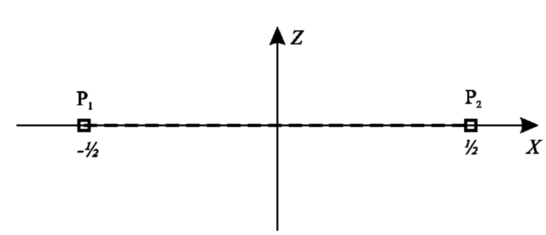

Fig. 4 Measurement path in the X–Z reference frame

Under the above assumptions the inverse dimensionless velocity, F_+, defined in Eq. 18, can be expanded as follows

$$\begin{aligned} F_+ &\simeq \frac{1}{UM + 1 - \frac{Z'^2_{F+}}{2} + O(Z'^4_{F+})} \\ &\simeq 1 - \left(UM - \frac{Z'^2_{F+}}{2} + O(Z'^4_{F+})\right) \\ &+ \left(UM - \frac{Z'^2_{F+}}{2} + O(Z'^4_{F+})\right)^2 \\ &- \left(UM - \frac{Z'^2_{F+}}{2} + O(Z'^4_{F+})\right)^3, \end{aligned} \tag{20}$$

and finally

$$F_+ \simeq 1 - MU + \frac{Z'^2_{F+}}{2} + M^2U^2 - M^3U^3 - MUZ'^2_{F+} + O(M^4), \tag{21}$$

where it has been considered that $Z_F = O(M)$. To get an expression of $F_+ = F_+(Z_+, Z'_+, X)$, Z'_{F+} should be expressed in terms of the trajectory shape $Z_+(x)$, which is obtained from Eqs. 14, 15

$$\begin{aligned} Z'_+ &= \frac{dZ_+}{dX} = \frac{w_p}{u_p} = \frac{Z'_{F+} + MW\sqrt{1 + Z'^2_{F+}}}{1 + MU\sqrt{1 + Z'^2_{F+}}} \simeq Z'_{F+}(1 - MU) \\ &+ MW - M^2WU + O(M^3), \end{aligned} \tag{22}$$

and solving for Z'_{F+}

$$Z'_{F+} \simeq \frac{Z'_+ - MW + M^2UW}{1 - MU} \simeq Z'_+(1 + MU) - MW + O(M^3) \tag{23}$$

Note that there are no M^2 order terms. On the other hand, as Z'_{F+} appears to be taken at the second power in (21) there is no need to retain smaller order terms in (23). By substituting (23) in (21) it is found that

$$\begin{aligned} F_+ &= 1 - MU + \frac{Z'^2_+}{2} - MWZ'_+ + M^2\left(U^2 + \frac{W^2}{2}\right) \\ &+ M^2UWZ'_+ - M^3U(U^2 + W^2), \end{aligned} \tag{24}$$

where terms of order M^4 and smaller have been neglected.

Substituting (24) in (10) instead of f, the following second order differential equation is obtained

$$Z''_+ + M(U_Z - W_X) + M^2(UW_X + WU_X - 2UU_Z - WW_Z) = 0 \tag{25}$$

In the limit $M \ll 1$, the fluid can be considered incompressible and, in the case of two-dimensional flow, the continuity equation leads to $U_X = -W_Z$, thus giving rise to

$$Z''_+ + M(U_Z - W_X) + M^2[UW_X - 2(UU_Z + WW_Z)] = 0 \tag{26}$$

If the functions U, W and their derivatives with regard to X and Z are evaluated on the trajectory $(X, Z_+(X))$, finding the solution of Eq. 26 could be a very complex process. However, the process can be simplified if it is assumed that the lateral shifts from the straight line are small ($Z_+ \ll 1$). The trajectory can then be expressed in a series expansion as

$$Z_+ = Z_{+1}M + Z_{+2}M^2 + \cdots \tag{27}$$

The values of functions U, W, and their derivatives at (X, Z_+) can be expressed in asymptotic form as follows

$$\begin{aligned} U(X,Z) &\simeq U(X,0) + U_Z(X,0)Z_{+1}M = U_0 + U_{Z0}Z_{+1}M, \\ U_X &\simeq U_X(X,0) + U_{XZ}(X,0)Z_{+1}M = U_{X0} + U_{XZ0}Z_{+1}M, \\ U_Z &\simeq U_Z(X,0) + U_{ZZ}(X,0)Z_{+1}M = U_{Z0} + U_{ZZ0}Z_{+1}M, \\ W(X,Z) &\simeq W(X,0) + W_Z(X,0)Z_{+1}M = W_0 + W_{Z0}Z_{+1}M, \\ W_X &\simeq W_X(X,0) + W_{XZ}(X,0)Z_{+1}M = W_{X0} + W_{XZ0}Z_{+1}M, \\ W_Z &\simeq W_Z(X,0) + W_{ZZ}(X,0)Z_{+1}M = W_{Z0} + W_{ZZ0}Z_{+1}M, \end{aligned} \tag{28}$$

Substituting Eqs. 27 and 28 in Eq. 26, the identification of terms of the same order leads to two problems to determine Z_{+1} and Z_{+2}, placed below

$$\text{order } M^1 : Z''_{+1} + (U_{Z0} - W_{X0}) = 0 \tag{29}$$

$$\begin{aligned} \text{order } M^2 : Z''_{+2} &- 2(U_0U_{Z0} + W_0W_{Z0}) \\ &+ U_0W_{X0} + Z_{+1}(U_{ZZ0} - W_{XZ0}) = 0 \end{aligned} \tag{30}$$

A similar process can be applied to analyze the travel of the sound signal from P2 to P1. In this case

$$T_- = t_-\frac{c}{l} = \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{dX}{UM - \frac{1}{\sqrt{1+Z'^2_{F-}}}} = -\int_{-\frac{1}{2}}^{\frac{1}{2}} F_- dX \tag{31}$$

where

$$\begin{aligned} F_- &= -\left[1 + MU + \frac{Z'^2_-}{2} + MWZ'_- + M^2\left(U^2 + \frac{W^2}{2}\right)\right. \\ &\left. + M^2UWZ'_- + M^3(U^3 + UW^2)\right]. \end{aligned} \tag{32}$$

From the Euler equation (10) applied to F_- and with the series expansion of Z_- the problems corresponding to backward travel are obtained

$$M^1 : Z''_{-1} - (U_{Z0} - W_{X0}) = 0 \tag{33}$$

$$M^2 : Z''_{-2} - 2(U_0 U_{Z0} + W_0 W_{Z0}) + U_0 W_{X0} - Z_{-1}(U_{ZZ0} - W_{XZ0}) = 0 \tag{34}$$

As was shown in Eqs. 29 and 33, at the first order the trajectory curvature is of order M and proportional to the component normal to the x, z plane of the rotational of the velocity field along the trajectory, which matches the result of Eq. 6 obtained by Landau and Lifshitz (1989). However, there is another contribution, of order M^2, given by Eqs. 30 and 34 not considered in the above referenced work.

For a given velocity field along the measurement path, solutions for Eqs. 29, 30, 33 and 34 can be found, and therefore the minimum transit time between sensors P1 and P2 in both directions can be determined.

From these expressions, and taking into account the boundary conditions ($Z_+(\pm 1/2) = Z_-(\pm 1/2) = 0$) it is easily shown that $Z_{+1}(X) = -Z_{-1}(X) = Z_1(X)$, and $Z_{+2}(X) = Z_{-2}(X) = Z_2(X)$. Therefore the trajectories in both directions can be expressed as follows

$$Z_+ = Z_1 M + Z_2 M^2, \tag{35}$$

$$Z_- = -Z_1 M + Z_2 M^2, \tag{36}$$

These results will be used in the following section devoted to the analysis of the effect of the trajectory curvature on both the transit-time and the measured speed.

Note that in the case of a non-rotational velocity field, the M^1 order equations give

$$Z_{+1} = Z_{-1} = 0 \tag{37}$$

which could also be deduced from Landau and Lifshitz (1989) result (Eq. 6). The trajectory curvature, however, is not zero because it also includes second order terms Z_{+2} and Z_{-2}, which satisfy the relationships

$$Z''_{+2} = Z''_{-2} = 2W_0 W_{Z0} + U_0 U_{Z0} \tag{38}$$

as derived from Eq. 30, taking into account the non-rotational condition $U_{Z0} = W_{X0}$. Thus, the shift of the trajectory with regard to the straight line in non-rotational flows is of order M^2.

5 Transit time and measured velocity

Once the minimum time trajectory is known, it is possible to determine the transit time, and the influence on the measured speed of both the trajectory shift and the Mach number. The process involves the substitution of the functions associated with the velocity field (Eq. 28) and the solution for Z_+, Z_- in the functions F_+ and F_- given by Eqs. 24, 32 to obtain

$$F_+ = 1 - U_0 M + \left(U_0^2 + G_{+2}\right)M^2 - \left(U_0^3 - G_{+3}\right)M^3 \tag{39}$$

$$-F_- = 1 + U_0 M + \left(U_0^2 + G_{-2}\right)M^2 + \left(U_0^3 + G_{-3}\right)M^3 \tag{40}$$

where

$$G_{+2} = \frac{1}{2}\left(W_0 - Z'_{+1}\right)^2 - Z_{+1}U_{Z0} \tag{41}$$

$$G_{-2} = \frac{1}{2}\left(W_0 + Z'_{-1}\right)^2 + Z_{-1}U_{Z0} \tag{42}$$

$$G_{+3} = Z_{+1}\left(2U_0 U_{Z0} + W_0 W_{Z0} - Z'_{+1} W_{Z0}\right) + Z'_{+1}\left(Z'_{+2} + U_0 W_0\right) - Z_{+2}U_{Z0} - Z'_{+2}W_0 - U_0 W_0^2, \tag{43}$$

$$G_{-3} = Z_{-1}\left(2U_0 U_{Z0} + W_0 W_{Z0} + Z'_{-1} W_{Z0}\right) + Z'_{-1}\left(Z'_{-2} + U_0 W_0\right) + Z_{-2}U_{Z0} + Z'_{-2}W_0 + U_0 W_0^2. \tag{44}$$

In the G_i functions, all the terms that modify the pulse transit time are gathered as a result of considering that the total pulse velocity is given by Eqs. 14 and 15.

The previous formulation can be simplified taking into account two points:

(1) the symmetry relationships $Z_{+1} = -Z_{-1} = Z_1$ and $Z_{+2} = Z_{-2} = Z_2$, which imply that the influence of the M^2 term in the transit time is the same in both directions, that is, $G_{+2} = G_{-2} = G_2$, and
(2) the influence of the order M^3 term, which is the opposite in both directions $G_{+3} = -G_{-3} = G_3$.

Introducing these considerations, the expressions 39 and 40 can be written as follows

$$F_+ = 1 - U_0 M + \left(U_0^2 + G_2\right)M^2 - \left(U_0^3 + G_3\right)M^3 \tag{45}$$

$$-F_- = 1 + U_0 M + \left(U_0^2 + G_2\right)M^2 + \left(U_0^3 + G_3\right)M^3 \tag{46}$$

Both dimensionless transit times, T_+ and T_-, are determined by using Eqs. 17 and 31 with Eqs. 45 and 46

$$T_+ = 1 - \overline{U_0}M + \left(\overline{U_0^2} + \mathrm{I}_{G2}\right)M^2 - \left(\overline{U_0^3} + \mathrm{I}_{G3}\right)M^3 \tag{47}$$

$$T_- = 1 + \overline{U_0}M + \left(\overline{U_0^2} + \mathrm{I}_{G2}\right)M^2 + \left(\overline{U_0^3} + \mathrm{I}_{G3}\right)M^3 \tag{48}$$

where

$$\overline{U_0} = \int_{-\frac{1}{2}}^{\frac{1}{2}} U_0 \mathrm{d}X, \quad \overline{U_0^2} = \int_{-\frac{1}{2}}^{\frac{1}{2}} U_0^2 \mathrm{d}X, \quad \overline{U_0^3} = \int_{-\frac{1}{2}}^{\frac{1}{2}} U_0^3 \mathrm{d}X,$$

$$\mathrm{I}_{G2} = \int_{-\frac{1}{2}}^{\frac{1}{2}} G_2 \mathrm{d}X, \quad \mathrm{I}_{G3} = \int_{-\frac{1}{2}}^{\frac{1}{2}} G_3 \mathrm{d}X, \tag{49}$$

Note that $\bar{U}_0$ is the mean value of the wind speed component along the measurement path. From the Eq. 2 the measured speed U_M, in dimensionless form, is given by

$$U_M = \frac{u_M}{v_R} = \frac{l}{2v_R}\left(\frac{1}{t_+} - \frac{1}{t_-}\right) = \frac{1}{2M}\left(\frac{1}{T_+} - \frac{1}{T_-}\right) \tag{50}$$

The inverses of transit times are obtained from the Eqs. 47 and 48 expanding in power series of M

$$\begin{aligned}\frac{1}{T_+} = 1 + \overline{U_0}M + \left(\overline{U_0}^2 - \overline{U_0^2} - \mathrm{I}_{G2}\right)M^2 \\ + \left(\mathrm{I}_{G3} - 2\mathrm{I}_{G2}\overline{U_0} + \overline{U_0}^3 - 2\overline{U_0}\,\overline{U_0^2} + \overline{U_0^3}\right)M^3 + O(M)^4\end{aligned} \tag{51}$$

$$\begin{aligned}\frac{1}{T_-} = 1 - \overline{U_0}M + \left(\overline{U_0}^2 - \overline{U_0^2} - \mathrm{I}_{G2}\right)M^2 \\ - \left(\mathrm{I}_{G3} - 2\mathrm{I}_{G2}\overline{U_0} + \overline{U_0}^3 - 2\overline{U_0}\,\overline{U_0^2} + \overline{U_0^3}\right)M^3 + O(M)^4\end{aligned} \tag{52}$$

Substituting Eqs. 51 and 52 for Eq. 50 the measured speed is given by

$$U_M = \overline{U_0}\left(1 + C_2M^2\right) + O(M^4) \tag{53}$$

where

$$C_2 = C_{20} + C_{2s} \tag{54}$$

$$C_{20} = \overline{U_0}^2 - 2\overline{U_0^2} + \frac{\overline{U_0^3}}{\overline{U_0}} \tag{55}$$

$$C_{2s} = \frac{\mathrm{I}_{G3}}{\overline{U_0}} - 2\mathrm{I}_{G2} \tag{56}$$

The correction terms caused by the straight-line trajectory and by the shift of the trajectory are gathered in the coefficients C_{20} and C_{2s}, respectively. It can be shown that, if just the C_{20} term is considered, expression 53 matches the already published results concerning the propagation of an acoustic pulse along a straight line trajectory in a steady, non-uniform velocity field (Cuerva and Sanz-Andrés 2000).

As can be deduced from Eq. 53, the measured velocity U_M is the average value of the velocity component along the measurement path, $\overline{U_0}$, plus an additional M^2 order term, which depends on the velocity field and on the minimum time trajectories between P1 and P2 (forward and backward). Observe that the M^2 order terms in Eqs. 51 and 52 cancel each other and all terms decrease one order because in (50) they are divided by M.

6 Application to some elementary and paradigmatic flows

The aim of this section is to show the effect of the higher order terms and the trajectory shift on the measured speed. The cases considered are: (1) A one-directional horizontal shear flow, (2) A one-directional vertical shear flow, (3) A one-directional vertical shear flow, but combined with a uniform horizontal velocity field.

6.1 One-directional horizontal shear flow

Let us consider the horizontal shear flow shown in Fig. 5 defined by $u = u_0 + \mathrm{d}u/\mathrm{d}z$ (where u_0 and $\mathrm{d}u/\mathrm{d}z$ are constants), and $w = 0$. Using as the reference speed u_0, $v_R = u_0$, these expressions become, in dimensionless form

$$U = \mathrm{K}_Z Z + 1 \tag{57}$$

$$W = 0 \tag{58}$$

where

$$\mathrm{K}_Z = \frac{\mathrm{d}u}{\mathrm{d}z}\frac{l}{u_0} = \frac{\mathrm{d}U}{\mathrm{d}Z} \tag{59}$$

K_Z is the value of the rotor of the velocity field. The trajectories can be found by using Eqs. 29 and 30

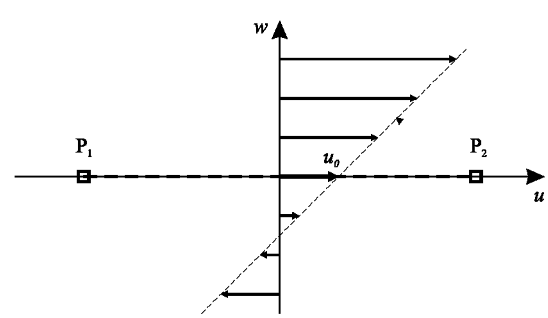

Fig. 5 Horizontal one-dimensional shear flow. $X = \pm 1/2$: acoustic path transducer positions

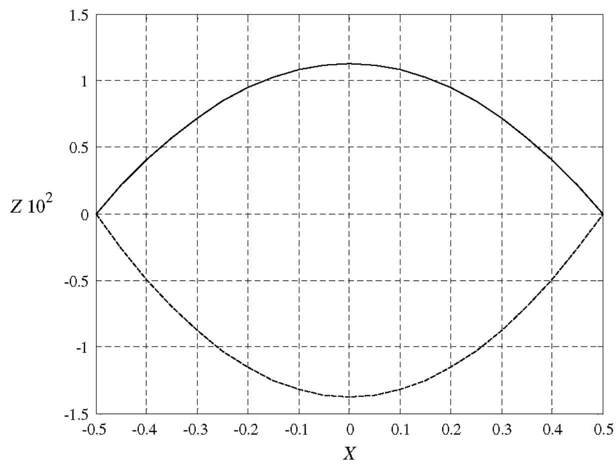

Fig. 6 Ultrasound pulse trajectories of minimum time: direction 1–2 (*solid line*), direction 2–1 (*dashed line*). $K_z = 1$, $M = 0.1$.

$$Z_1'' + K_Z = 0 \tag{60}$$

$$Z_2'' - 2K_Z = 0 \tag{61}$$

together with the boundary conditions $Z_+(\pm 1/2) = Z_-(\pm 1/2) = 0$, the solutions are

$$Z_1 = \frac{K_Z}{8}\left(1 - 4X^2\right) \tag{62}$$

$$Z_2 = -\frac{K_Z}{4}\left(1 - 4X^2\right) \tag{63}$$

Replacing both solutions in Eqs. 35 and 36, the pulse trajectories in both directions are obtained

$$Z_+ = \frac{K_Z}{8}\left(1 - 4X^2\right)\left(M - 2M^2\right) \tag{64}$$

$$Z_- = -\frac{K_Z}{8}\left(1 - 4X^2\right)\left(M + 2M^2\right) \tag{65}$$

Both trajectories are parabolas with the maximum deviation at $X = 0$, whose value depends on the parameters K_Z and M.

As it can be shown in Fig. 6, the amplitudes of the forward and backward pulse trajectories are different, because of the sign of the M^2 term. Once the pulse trajectories are determined, the coefficients (49) are obtained

$$\overline{U_0} = \overline{U_0^2} = \overline{U_0^3} = 1, \qquad I_{G2} = -\frac{K_Z^2}{24}, \qquad I_{G3} = -\frac{K_Z^2}{6} \tag{66}$$

and therefore $C_{20} = 0$. Generally speaking, if the velocity component along the measurement path is constant (as in this case), from Eq. 55 it is deduced that $C_{20} = 0$. It means that the correction related to the straight propagation disappears, and the only corrections are those caused by the shift of the trajectory.

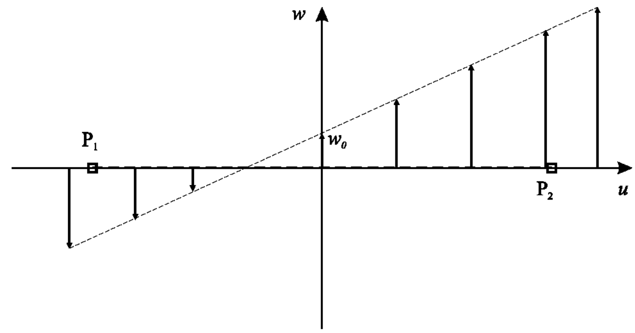

Fig. 7 Vertical one-dimensional shear flow. $X = \pm 1/2$: acoustic path sensor positions

Then, the transit times of the forward and backward pulses are, respectively,

$$T_+ = 1 - M + \left(1 - \frac{K_Z^2}{24}\right)M^2 - \left(1 - \frac{K_Z^2}{6}\right)M^3 \tag{67}$$

$$T_- = 1 + M + \left(1 - \frac{K_Z^2}{24}\right)M^2 + \left(1 - \frac{K_Z^2}{6}\right)M^3 \tag{68}$$

The M^2 order terms do not have any influence in the measured speed, as these terms have the same sign, and therefore they cancel out each other when the difference of inverses of times is calculated.

The measured speed is therefore

$$U_M = 1 - \frac{K_Z^2}{12}M^2 + O\left(M^4\right) \tag{69}$$

Observe that the speed measured by the ultrasonic measurement path is always slightly lower than the value of the speed averaged along the straight line, with the reference speed u_0, assumed to be the nominal value of the measurement.

6.2 One-directional vertical shear flow

Let us consider the flow shown in Fig. 7, defined as $u = 0$, $w = w_0 + \mathrm{d}w/\mathrm{d}z$, where w_0 and $\mathrm{d}w/\mathrm{d}z$ are constants.

Considering $v_R = w_0$ and following a similar procedure to that used in the previous section, the velocity field can be expressed as follows

$$U = 0 \tag{70}$$

$$W = K_X X + 1, \quad K_X = \frac{\mathrm{d}w}{\mathrm{d}x}\frac{l}{w_0} = \frac{\mathrm{d}W}{\mathrm{d}X} \tag{71}$$

By using (29) and (30), the trajectories are now

$$Z_+ = -\frac{K_Z}{8}\left(1-4X^2\right)M \tag{72}$$

$$Z_- = \frac{K_Z}{8}\left(1-4X^2\right)M \tag{73}$$

Note that the solution of the M^2 order problem (30), which defines Z_2, leads to $Z_2 = 0$. As in the previous section, both trajectories are parabolas with maximum deviation at $X = 0$, but unlike the previous case now both trajectories are symmetric with regard to the X axis. The transit times are identical

$$T_+ = T_- = 1 + \frac{1}{2}M^2 \tag{74}$$

The transit times of the pulses are both larger than in the case of an unperturbed flow, but one effect cancels out the other one, and therefore the measured speed is not affected by the trajectory shift (the measured velocity will be $U_M = 0$). Observe that the velocity level w_0 does not affect the results.

6.3 One-directional vertical shear in an otherwise uniform horizontal flow

Let us consider a modification of the previous flow by adding a uniform horizontal speed u_0, as shown in Fig. 8. Considering $v_R = u_0$, the dimensionless velocity field can be expressed as

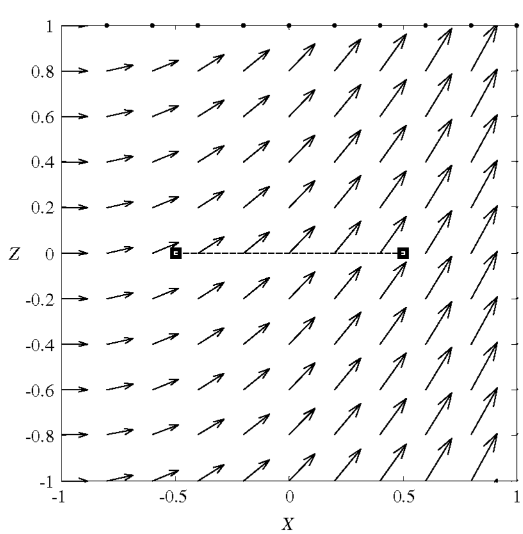

Fig. 8 Vertical one-dimensional shear plus uniform horizontal velocity flow. $K_x = V_0 = 1$. $X = \pm 1/2$: acoustic path sensor positions

$$\begin{aligned} U &= 1, \\ W &= K_X X + V_0, \end{aligned} \tag{75}$$

where

$$K_X = \frac{dw}{dx}\frac{l}{u_0} = \frac{dW}{dX}, \quad V_0 = \frac{w_0}{u_0} \tag{76}$$

The trajectories are now based on Z_1 and Z_2, obtained from Eqs. 29 and 30

$$Z_1 = -\frac{K_X}{8}\left(1-4X^2\right) \tag{77}$$

$$Z_2 = \frac{K_X}{8}\left(1-4X^2\right) \tag{78}$$

and therefore the trajectories in both directions are given by

$$Z_+ = -\frac{K_X}{8}\left(1-4X^2\right)\left(M-M^2\right) \tag{79}$$

$$Z_- = \frac{K_X}{8}\left(1-4X^2\right)\left(M+M^2\right) \tag{80}$$

The respective transit times are

$$T_+ = 1 - M + \left(1+\frac{V_0^2}{2}\right)M^2 - \left(1-\frac{K_X^2}{6}-V_0^2\right)M^3 \tag{81}$$

$$T_- = 1 + M + \left(1+\frac{V_0^2}{2}\right)M^2 + \left(1-\frac{K_X^2}{6}-V_0^2\right)M^3 \tag{82}$$

The term M^2 gives a positive contribution to the transit times in both directions, and therefore this term does not affect the measured speed. However, as the M^3 order terms have different signs, they do not cancel out when the inverses of time are subtracted, and thus they are responsible for differences in measured speed. As in Eq. 50 the result is divided by M, and the terms of M^3 order give rise to the M^2 order terms in U_M

$$U_M = 1 - \left(\frac{K_X^2}{6} + 2V_0^2\right)M^2 \tag{83}$$

The M^2 order correction term is exclusively due to the trajectory curvature, as the horizontal velocity component is constant and therefore $C_{20} = 0$. As previously obtained in the one-directional horizontal shear flow, the correction term is negative, and thus the measured speed will be lower than the "true value" (which is the average along the straight trajectory). In Fig. 9 the coefficient

$$C_{2s} = -\left(\frac{K_X^2}{6} + 2V_0^2\right) \tag{84}$$

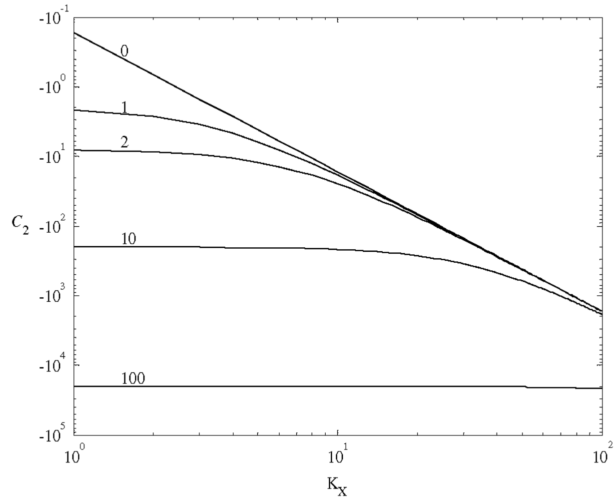

Fig. 9 Variation of the correction coefficient, C_{2S}, with the rotational value K_x. *Figures* indicate the value of the transversal speed at the centre of the measurements path, V_0

is shown as a function of the velocity rotor component normal to the plane, K_X, and the speed ratio V_0. Observe that, although the correction is C_{2s} times M^2, even if $M \ll 1$, the correction may be important.

7 Conclusions

In this paper, a mathematical model of the measurement process performed by an ultrasonic anemometer, based on the acoustic geometric theory, developed in order to determine the flow velocity component along the measurement path, is presented. The main contribution of the paper is the consideration of the impact of the shift of the acoustic pulse trajectory (with regard to the straight line) on the measured speed, something which had not been taken into account in previous published works. An approximate solution of the problem, the velocity measured by the anemometer, has been obtained by asymptotic techniques, based on the small value of the Mach number, M, and the shift of the trajectory. This way, an analytic solution has been obtained, which allows the analysis and identification of not only the influence of the several parameters involved, but of the flow characteristics as well. It can be shown that if the flow is rotational, the trajectory shift is of M order (as pointed out by Landau and Lifshitz 1989) and if it is non-rotational, the shift is of order M^2.

Regarding the forward and backward propagation of the pulse, the odd order contributions to the trajectory shift are anti-symmetric, while the even order contributions are symmetric. This result helps in further simplifying the calculation of the final results.

The "measured" speed (the speed measured by the anemometer as determined by the present model) is obtained and compared to the "true" speed, that is, the average velocity along the measurement path. The "measured" speed contains two factors: 1) the average speed or "true" speed, and 2) a term $1 + M^2C_2$, which includes the correction term M^2C_2. In its own side, the correction C_2 is composed of two terms: (1) the contribution of the velocity field along the straight line, C_{20}, which only depends on the uniformity of the path-wise velocity component, U_0 (and is zero if U_0 is uniform), and (2) the contribution of the trajectory shift effects, C_{2s}, including the variation along the measurement path of the value of the velocity components and their longitudinal and transversal derivatives.

In order to clarify the results obtained, some typical examples of paradigmatic and elementary flows have been considered: (1) horizontal one-dimensional shear flow, (2) vertical one-dimensional shear flow, and (3) vertical one-dimensional shear flow plus a uniform horizontal speed. The "measured" velocity has been obtained and analyzed. In the first case, a negative correction (proportional to the rotational value squared) appears, meaning that the "measured" speed is lower than the "true" speed. In the second case, the fluid flow does not have an influence on the "measured" speed, because the transit times in both directions are influenced in the same way. In the third case, a negative correction appears (as in the first case) although in this case in addition to the rotational value squared, the value of the transverse velocity component squared is involved as well.

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